

# A 6-DIMENSIONAL SIMPLY CONNECTED COMPLEX AND SYMPLECTIC MANIFOLD WITH NO KÄHLER METRIC

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**ABSTRACT.** We construct a simply connected compact manifold which has complex and symplectic structures but does not admit Kähler metric, in the lowest possible dimension where this can happen, that is, dimension 6. Such a manifold is automatically formal and has even odd-degree Betti numbers but it does not satisfy the Lefschetz property for any symplectic form.

## 1. INTRODUCTION

A Kähler manifold  $(M, J, \omega)$  is a smooth manifold  $M$  of dimension  $2n$  endowed with an integrable almost complex structure  $J$  and a symplectic form  $\omega$  such that  $g(X, Y) = \omega(X, JY)$  defines a Riemannian metric, called *Kähler metric*. In order to check that a compact manifold does not carry any Kähler metric, one can use a collection of known topological obstructions to the existence of such a structure: theory of Kähler groups, evenness of odd-degree Betti numbers, Lefschetz property or the formality of the rational homotopy type (see [1, 7, 23]).

If  $M$  is a compact Kähler manifold, then it has a complex and a symplectic structure. However, the converse is not true. The first example of a compact manifold admitting complex and symplectic structures but no Kähler metric is the Kodaira-Thurston manifold [16, 21]. This 4-manifold is not simply connected (it is actually a nilmanifold) hence the fundamental group plays a key role in this property. The classification of complex and symplectic nilmanifolds of dimension 6 was given by Salamon in [20]. Generalizations to higher dimension  $2n \geq 6$  of the Kodaira-Thurston manifold are the generalized Iwasawa manifolds considered in [6]. Such manifolds have complex and symplectic structures but carry no Kähler metric. Note that, in dimension 2, every oriented surface admits a Kähler metric.

If one restricts attention to manifolds with trivial fundamental group, then every complex manifold of real dimension 4 admits a Kähler structure. Indeed, by the Enriques-Kodaira classification [16], if  $M$  is a complex surface whose first Betti number  $b_1$  is even (this holds in particular when  $b_1 = 0$ ), then  $M$  is deformation equivalent to a Kähler surface (see also [2, Theorem 3.1, page 144] for a direct proof of this fact). We point out that Gompf [13] has constructed the first examples of simply connected compact symplectic but not complex 4-manifolds. Also Fintushel and Stern [12] have given a family of simply connected symplectic 4-manifolds not admitting complex structures (the latter was proved by Park [19]).

In dimensions higher than 4, we have the following results. The first examples of simply connected compact symplectic non-Kählerian manifolds were given in dimension 6 by Gompf in the aforementioned paper [13] and in dimension  $\geq 10$  by McDuff in [17] (these examples are not known to admit complex structures). Fine and Panov in [10] (see also [11]) have produced

simply connected symplectic 6-manifolds with  $c_1 = 0$  which do not have a compatible complex structure (but it is not known if they admit Kähler structures). Furthermore, Guan in [14] constructed the first family of simply connected, compact and holomorphic symplectic non-Kählerian manifolds of (real) dimension  $4n \geq 8$ . On the other hand, the first and third authors have proved [3] that the 8-dimensional manifold  $X$  constructed in [9] is an example of a simply connected, symplectic and complex manifold which does not admit a Kähler structure (since it is not formal). For higher dimensions  $2n = 8 + 2k$ ,  $k \geq 1$ , one can take  $X \times \mathbb{CP}^k$ . This is simply connected, complex and symplectic but not Kähler. Thus, a natural question arises:

*Does there exist a 6-dimensional simply connected, compact, symplectic and complex manifold which does not admit Kähler metrics?*

In this paper we answer this question in the affirmative by proving the following result:

**Theorem 1.1.** *There exists a 6-dimensional, simply connected, compact, symplectic and complex manifold which carries no Kähler metric.*

In order to construct such an example, we start with a 6-dimensional nilmanifold  $M$  admitting both a complex structure  $J$  and a symplectic structure  $\omega$ . Then we quotient it by a finite group preserving  $J$  and  $\omega$  to obtain a simply connected, 6-dimensional orbifold  $\widehat{M}$  with an orbifold complex structure  $\widehat{J}$  and an orbifold symplectic form  $\widehat{\omega}$ . By Hironaka Theorem [15], there is a complex resolution  $(\widetilde{M}_c, \widetilde{J})$  of  $(\widehat{M}, \widehat{J})$ . As in [5], we resolve symplectically the singularities of  $(\widehat{M}, \widehat{\omega})$  to obtain a smooth symplectic 6-manifold  $(\widetilde{M}_s, \widetilde{\omega})$ . However, in our situation, the singular locus of the orbifold  $\widehat{M}$  does not consist only of a discrete set of points, in contrast with [5]. For a complex and symplectic orbifold, we provide conditions under which the complex and the symplectic resolution of singularities are diffeomorphic (Theorem 3.1). Using this we prove that the resolutions  $\widetilde{M}_c$  and  $\widetilde{M}_s$  are diffeomorphic. Thus,  $\widetilde{M} = \widetilde{M}_c$  is not only a complex manifold but also a symplectic one.

Next, we show that  $\widetilde{M}$  is simply connected (Proposition 6.1), this resulting from the careful choice of the action of the finite group on  $M$ . Since  $\widetilde{M}$  is a 6-dimensional simply connected compact manifold, then  $b_1(\widetilde{M}) = 0$ , and  $b_3(\widetilde{M})$  is even by Poincaré duality. Also  $\widetilde{M}$  is automatically formal by [8, Theorem 3.2]. Therefore, to ensure that  $\widetilde{M}$  does not carry any Kähler metric, we use the Lefschetz property; more precisely, we prove that the map  $L_{[\Omega]}: H^2(\widetilde{M}) \rightarrow H^4(\widetilde{M})$  given by the cup product with  $[\Omega]$  is not an isomorphism for any possible symplectic form  $\Omega$ . Again the choice of nilmanifold  $M$  and finite group action makes possible to have a non-zero  $[\beta] \in H^2(\widetilde{M})$  such that  $[\beta] \wedge [\alpha_1] \wedge [\alpha_2] = 0$  for every  $[\alpha_1], [\alpha_2] \in H^2(\widetilde{M})$ , which gives the result.

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## 2. ORBIFOLDS

**Definition 2.1.** A (smooth)  $n$ -dimensional orbifold is a Hausdorff, paracompact topological space  $X$  endowed with an atlas  $\mathcal{A} = \{(U_p, \tilde{U}_p, \Gamma_p, \varphi_p)\}$  of orbifold charts, that is  $U_p \subset X$  is a neighbourhood of  $p \in X$ ,  $\tilde{U}_p \subset \mathbb{R}^n$  an open set,  $\Gamma_p \subset \mathrm{GL}(n, \mathbb{R})$  a finite group acting on  $\tilde{U}_p$ , and  $\varphi_p: \tilde{U}_p \rightarrow U_p$  is a  $\Gamma_p$ -invariant map with  $\varphi_p(0) = p$ , inducing a homeomorphism  $\tilde{U}_p/\Gamma_p \cong U_p$ .

The charts are compatible in the following sense: if  $q \in U_q \cap U_p$ , then there exist a connected neighbourhood  $V \subset U_q \cap U_p$  and a diffeomorphism  $f: \varphi_p^{-1}(V)_0 \rightarrow \varphi_q^{-1}(V)$ , where  $\varphi_p^{-1}(V)_0$  is the connected component of  $\varphi_p^{-1}(V)$  containing  $q$ , such that  $f(\sigma(x)) = \rho(\sigma)(f(x))$ , for any  $x$ , and  $\sigma \in \mathrm{Stab}_{\Gamma_p}(q)$ , where  $\rho: \mathrm{Stab}_{\Gamma_p}(q) \rightarrow \Gamma_q$  is a group isomorphism.

For each  $p \in X$ , let  $n_p = \#\Gamma_p$  be the order of the orbifold point (if  $n_p = 1$  the point is smooth, also called non-orbifold point). The singular locus of the orbifold is the set  $S = \{p \in X \mid n_p > 1\}$ . Therefore  $M - S$  is a smooth  $n$ -dimensional manifold. The singular locus  $S$  is stratified: if we write  $S_k = \{p \mid n_p = k\}$ , and consider its closure  $\overline{S_k}$ , then  $\overline{S_k}$  inherits the structure of an orbifold. In particular  $S_k$  is a smooth manifold, and the closure consists of some points of  $S_{kl}$ ,  $l \geq 2$ .

We say that the orbifold is locally oriented if  $\Gamma_p \subset \mathrm{GL}_+(n, \mathbb{R})$  for any  $p \in X$ . As  $\Gamma_p$  is finite, we can choose a metric on  $\tilde{U}_p$  such that  $\Gamma_p \subset \mathrm{SO}(n)$ . An element  $\sigma \in \Gamma_p$  admits a basis in which it is written as  $\sigma = \mathrm{diag} \left( \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{pmatrix}, \dots, \begin{pmatrix} \cos \theta_r & -\sin \theta_r \\ \sin \theta_r & \cos \theta_r \end{pmatrix}, 1, \dots, 1 \right)$ , for  $\theta_1, \dots, \theta_r \in (0, 2\pi)$ . In particular, the set of points fixed by  $\sigma$  is of codimension  $2r$ . Therefore the set of singular points  $S \cap U_p$  is of codimension  $\geq 2$ , and hence  $X - S$  is connected (if  $X$  is connected). Also we say that the orbifold  $X$  is oriented if it is locally oriented and  $X - S$  is oriented.

A natural example of orbifold appears when we take a smooth manifold  $M$  and a finite group  $\Gamma$  acting on  $M$  effectively. Then  $\widehat{M} = M/\Gamma$  is an orbifold. If  $M$  is oriented and the action of  $\Gamma$  preserves the orientation, then  $\widehat{M}$  is an oriented orbifold. Note that for every  $\widehat{p} \in \widehat{M}$ , the group  $\Gamma_{\widehat{p}}$  is the stabilizer of  $p \in M$ , with  $\widehat{p} = \widehat{\pi}(p)$  under the natural projection  $\widehat{\pi}: M \rightarrow \widehat{M}$ .

**Definition 2.2.** A complex orbifold is a  $2n$ -dimensional orbifold  $X$  whose orbifold charts have  $\tilde{U}_p \subset \mathbb{C}^n$ ,  $\Gamma_p \subset \mathrm{GL}(n, \mathbb{C})$ , and in the compatibility of charts the maps  $f$  are biholomorphisms. Note that  $X$  is automatically oriented.

If  $M$  is a complex manifold and  $\Gamma$  is a finite group acting effectively on  $M$  by biholomorphisms, then  $\widehat{M} = M/\Gamma$  is a complex orbifold.

The complex structure of a complex orbifold  $X$  can be given by the orbifold  $(1, 1)$ -tensor  $J$  with  $J^2 = -\mathrm{Id}$ . This is given by tensors  $J_p$  on each  $\tilde{U}_p$  defining the complex structure, which are  $\Gamma_p$ -equivariant, for each  $p \in X$ , and which agree under the functions  $f$  defining the compatibility of charts.

**Definition 2.3.** A complex resolution of a complex orbifold  $(X, J)$  is a complex manifold  $\tilde{X}$  together with a holomorphic map  $\pi: \tilde{X} \rightarrow X$  which is a biholomorphism  $\tilde{X} - E \rightarrow X - S$ , where  $S \subset X$  is the singular locus and  $E = \pi^{-1}(S)$  is the exceptional locus.

Let  $X$  be an orbifold. An orbifold  $k$ -form  $\alpha$  consists of a collection of  $k$ -forms  $\alpha_p$  on each  $\tilde{U}_p$  which are  $\Gamma_p$ -equivariant and that match under the compatibility maps between different charts.

**Definition 2.4.** *A symplectic orbifold  $(X, \omega)$  consists of a  $2n$ -dimensional oriented orbifold  $X$  and an orbifold 2-form  $\omega$  such that  $d\omega = 0$  and  $\omega^n > 0$  everywhere.*

If  $M$  is a symplectic manifold and  $\Gamma$  is a finite group acting effectively on  $M$  by symplectomorphisms, then  $\widehat{M} = M/\Gamma$  is a symplectic orbifold.

**Definition 2.5.** *A symplectic resolution of a symplectic orbifold  $(X, \omega)$  consists of a smooth symplectic manifold  $(\tilde{X}, \tilde{\omega})$  and a map  $\pi: \tilde{X} \rightarrow X$  such that:*

- $\pi$  is a diffeomorphism  $\tilde{X} - E \rightarrow X - S$ , where  $S \subset X$  is the singular locus and  $E = \pi^{-1}(S)$  is the exceptional locus.
- $\tilde{\omega}$  and  $\pi^*\omega$  agree in the complement of a small neighbourhood of  $E$ .

### 3. DESINGULARIZATION OF ORBIFOLD POINTS

In this section we suppose that  $X$  is an oriented orbifold whose singular locus  $S$  consists of a discrete set of points. Assume that  $X$  admits a complex structure  $J$  and a symplectic structure  $\omega$ . Therefore we have a complex orbifold  $(X, J)$  and a symplectic orbifold  $(X, \omega)$ .

It is well-known that  $(X, J)$  admits a complex resolution  $(\tilde{X}_c, \tilde{J})$  by Hironaka's desingularization [15]. Also, the symplectic orbifold  $(X, \omega)$  admits a symplectic resolution  $(\tilde{X}_s, \tilde{\omega})$  by Theorem 3.3 in [5]. We want to compare the two resolutions.

First, let us look at the complex resolution of  $(X, J)$ . Consider  $p \in S$ , and let  $U_p = \tilde{U}_p/\Gamma_p$  be an orbifold neighbourhood. Recall that we denote  $\varphi_p: \tilde{U}_p \rightarrow U_p$  the quotient map. By definition of complex orbifold,  $\tilde{U}_p \subset \mathbb{C}^n = \mathbb{R}^{2n}$  and  $\Gamma_p \subset \text{GL}(n, \mathbb{C})$ . As  $\Gamma_p$  is a finite group, we can choose a Kähler metric invariant by  $\Gamma_p$ . With a linear change of variables, we can transform the Kähler metric into standard form. That is, we can suppose that there is an inclusion

$$(1) \quad \iota: \Gamma_p \hookrightarrow \text{U}(n).$$

Shrinking  $\tilde{U}_p$  if necessary, we can assume that  $\tilde{U}_p = B_\epsilon(0)$ , for some  $\epsilon > 0$ .

Consider now an algebraic resolution of the singularity of  $Y = \mathbb{C}^n/\Gamma_p$ , provided by [15]. Denote it  $\pi: \tilde{Y} \rightarrow Y$ , and let  $E = \pi^{-1}(p)$  be the exceptional locus. Write  $B = B_\epsilon(0)/\Gamma_p$  and  $\tilde{B} = \pi^{-1}(B)$ . The complex resolution is defined as the smooth manifold

$$\tilde{X}_c = (X - \{p\}) \cup_\pi \tilde{B},$$

where we identify with the map  $\pi: \tilde{B} - E \rightarrow B - \{p\} = U_p - \{p\}$ . This has a natural complex structure since  $\pi$  is a biholomorphism.

Now we move to the construction of the symplectic resolution of  $(X, \omega)$ , as done in [5]. For  $p \in S$ , take an orbifold neighbourhood  $U'_p = \tilde{U}'_p/\Gamma'_p$ , with  $\varphi'_p: \tilde{U}'_p \rightarrow U'_p$ . By the equivariant Darboux theorem, there is a  $\Gamma'_p$ -equivariant symplectomorphism  $(\tilde{U}'_p, \omega_p) \cong (V, \omega_0)$ , where  $V \subset \mathbb{R}^{2n}$  is an open set, and  $\omega_0$  is the standard symplectic form (shrinking  $\tilde{U}'_p$  if necessary). So without loss of generality, we can assume that  $\tilde{U}'_p \subset (\mathbb{R}^{2n}, \omega_0)$ , where  $\omega_0$  is the standard

symplectic form, and  $\Gamma'_p \subset \mathrm{Sp}(2n, \mathbb{R})$ . As  $\Gamma'_p$  is a finite group, and  $\mathrm{U}(n) \subset \mathrm{Sp}(2n, \mathbb{R})$  is the maximal compact subgroup, we can choose a complex structure  $J$  on  $\mathbb{R}^{2n}$  such that the pair  $(J, \omega_0)$  determines a Kähler metric, which is invariant by  $\Gamma'_p$ . We perform a linear change of variables, which transforms the complex structure into standard form (so  $\tilde{U}'_p$  has the standard Kähler structure). Equivalently, we can suppose that there is an inclusion

$$(2) \quad \iota': \Gamma'_p \hookrightarrow \mathrm{U}(n).$$

Shrinking  $\tilde{U}'_p$  if necessary, we can assume that  $\tilde{U}'_p = B_{\epsilon'}(0)$ , for some  $\epsilon' > 0$ .

Consider an algebraic resolution of singularities of  $Y' = \mathbb{C}^n / \Gamma'_p$ , say  $\pi': \tilde{Y}' \rightarrow Y'$ , and let  $E' = (\pi')^{-1}(p)$  be the exceptional locus. Write  $B' = B_{\epsilon'}(0) / \Gamma'_p$  and  $\tilde{B}' = (\pi')^{-1}(B')$ . The symplectic resolution is defined as the smooth manifold

$$\tilde{X}_s = (X - \{p\}) \cup_{\pi'} \tilde{B}',$$

where  $\tilde{B}' - E'$  and  $B' - \{p\} = U'_p - \{p\}$  are identified by  $\pi'$ . This has a symplectic structure that is constructed by gluing the symplectic structure of  $X - \{p\}$  and the Kähler form of  $\tilde{B}'$  by a cut-off process, as done in Theorem 3.3 of [5].

Now we are going to compare  $\tilde{X}_c$  and  $\tilde{X}_s$ . First note that for  $p \in S$ , we have  $\Gamma_p \cong \Gamma'_p$ . This follows from  $\Gamma_p \cong \pi_1(B - \{p\})$  and  $\Gamma'_p \cong \pi_1(B' - \{p\})$ , and the fact that  $B, B'$  are homeomorphic. So we shall denote  $\Gamma'_p = \Gamma_p$  henceforth. We have the following result.

**Theorem 3.1.** *If one can arrange that the inclusions  $\iota$  and  $\iota'$ , given by (1) and (2), respectively, are such that  $\iota = \iota'$  for every singular point  $p \in S$ , then there is a diffeomorphism  $\tilde{X}_c \cong \tilde{X}_s$ , which is the identity outside a small neighbourhood of the exceptional loci. In particular,  $\tilde{X}_c$  admits both complex and symplectic structures.*

*Proof.* The key point is obviously that if  $\iota = \iota'$ , then  $Y' = Y$ , so we can take  $\tilde{Y}' = \tilde{Y}$  and  $\pi' = \pi$  in the constructions above.

We fix a point  $p \in S$ , and construct the required isomorphism in a neighbourhood of the exceptional locus over that point. Consider the map (reducing  $\epsilon > 0$  if necessary)

$$f = (\varphi'_p)^{-1} \circ \varphi_p: B_\epsilon(0) = \tilde{U}_p \rightarrow B_{\epsilon'}(0) = \tilde{U}'_p;$$

$f$  is  $\Gamma_p$ -equivariant and an open embedding (it might fail to be surjective) with  $f(0) = 0$ . We shall construct a map  $F: B_\epsilon(0) \rightarrow B_{\epsilon'}(0)$  such that

- $F = \mathrm{Id}$  in a small ball  $B_{0.2\epsilon}(0)$ ,
- $F = f$  outside a slightly bigger ball  $B_{0.9\epsilon}(0)$ ,
- $F$  is a  $\Gamma_p$ -equivariant diffeomorphism onto its image.

This gives a diffeomorphism  $F: \tilde{X}_c \rightarrow \tilde{X}_s$ , defined by  $F$  on  $B_\epsilon(0) / \Gamma_p - \{p\}$ , extended by the identity on  $\pi^{-1}(B_{0.2\epsilon}(0) / \Gamma_p)$ , and also by the identity on  $X - \pi^{-1}(B_{0.9\epsilon}(0) / \Gamma_p)$ .

Write  $f(x) = L(x) + R(x)$ , where  $L$  is the linear part and  $|R(x)| \leq C|x|^2$ , for some constant  $C > 0$ . Both these maps are  $\Gamma_p$ -equivariant. Take a smooth, non-decreasing function  $\rho_1: [0, \epsilon] \rightarrow [0, 1]$  such that  $\rho_1(t) = 0$  for  $t \in [0, 0.8\epsilon]$  and  $\rho_1(t) = 1$  for  $t \in [0.9\epsilon, 1]$ . Consider  $g(x) = L(x) + \rho_1(|x|)R(x)$ . Then,  $g(x) = L(x)$  for  $|x| \leq 0.8\epsilon$ ,  $g(x) = f(x)$  for  $|x| \geq 0.9\epsilon$ , and  $g(x)$  is  $\Gamma_p$ -equivariant because  $\Gamma_p \subset \mathrm{SO}(2n)$ . Also

$$dg(x) - L = \rho'_1(|x|)R(x)d|x| + \rho_1(|x|)dR(x).$$

Using that  $|\rho'_1(t)| \leq C/\epsilon$  and  $|dR(x)| \leq C|x|$  (we denote by  $C > 0$  uniform constants, that can vary from line to line) we have that  $|dg(x) - L| \leq C|x|$ . For  $\epsilon > 0$  small enough, we have that  $g$  is a diffeomorphism onto its image.

For the next step, take the linear map  $L: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ . We can choose orthonormal (oriented) basis in both origin and target so that  $L = \text{diag}(\lambda_1, \dots, \lambda_{2n})$ , where  $\lambda_i > 0$  are real numbers (the first vector of the basis is a unitary vector  $e_1$  such that  $|L(e_1)|$  is maximized; then  $L$  maps  $\langle e_1 \rangle^\perp$  to  $\langle L(e_1) \rangle^\perp$ , and we proceed inductively). Consider the map

$$h(x) = \begin{cases} x, & |x| \leq 0.4\epsilon, \\ x + \rho_2 \left( \left( \frac{|x| - 0.4\epsilon}{0.3\epsilon} \right)^\alpha \right) (L(x) - x), & 0.4\epsilon \leq |x| \leq 0.7\epsilon, \\ g(x), & |x| \geq 0.7\epsilon, \end{cases}$$

where  $\rho_2: [0, 1] \rightarrow [0, 1]$  is smooth non-decreasing with  $\rho_2(t) = 0$  for  $t \in [0, \frac{1}{3}]$ , and  $\rho_2(t) = 1$  for  $t \in [\frac{2}{3}, 1]$ . Here  $\alpha > 0$  is a constant to be fixed soon.

Clearly  $h$  is  $\Gamma_p$ -equivariant,  $h(x) = f(x)$  off  $B_{0.9\epsilon}(0)$ , and  $h(x) = x$  in  $B_{0.4\epsilon}(0)$  (but beware, we have chosen different coordinates on the origin  $\mathbb{R}^{2n}$  and the target  $\mathbb{R}^{2n}$ , so  $h$  is not the identity in the ball). The map  $h$  is  $C^\infty$  because for  $0.4\epsilon \leq |x| \leq 0.5\epsilon$  we have also  $h(x) = x$ . Let us see that  $h$  is a diffeomorphism onto its image. It only remains to see this for  $0.5\epsilon \leq |x| \leq 0.7\epsilon$ . Write  $y = h(x)$ , so in our coordinates  $y_i = x_i + \rho_2(u)(\lambda_i - 1)x_i$ , with  $u = \left( \frac{|x| - 0.4\epsilon}{0.3\epsilon} \right)^\alpha$ . Then,

$$dy_i = (1 + (\lambda_i - 1)\rho_2(u)) dx_i + (\lambda_i - 1)\rho'_2(u) \frac{\alpha}{0.3\epsilon} \left( \frac{|x| - 0.4\epsilon}{0.3\epsilon} \right)^{\alpha-1} x_i \gamma$$

with  $\gamma = d|x| = \frac{1}{|x|} \sum x_j dx_j$ . Write  $\delta_i = (1 + (\lambda_i - 1)\rho_2(u))$ , so  $\delta_i$  takes values between 1 and  $\lambda_i$ . We compute

$$\begin{aligned} dy_1 \wedge \dots \wedge dy_n &= \delta_1 \dots \delta_n dx_1 \wedge \dots \wedge dx_n + \\ &+ \sum \delta_1 \dots \hat{\delta}_i \dots \delta_n \frac{(\lambda_i - 1)\rho'_2(u)\alpha x_i}{0.3\epsilon} \left( \frac{|x| - 0.4\epsilon}{0.3\epsilon} \right)^{\alpha-1} dx_1 \wedge \dots \wedge \overset{(i)}{\gamma} \wedge \dots \wedge dx_n \\ &= \delta_1 \dots \delta_n \left( 1 + \alpha \sum \frac{(\lambda_i - 1)\rho'_2(u)(|x| - 0.4\epsilon)^{\alpha-1} x_i^2}{|x|\delta_i(0.3\epsilon)^\alpha} \right) dx_1 \wedge \dots \wedge dx_n. \end{aligned}$$

In the sum, the numerator is bounded above by  $C(0.3\epsilon)^{\alpha+1}$  and the denominator is bounded below by  $C^{-1}(0.3\epsilon)^{\alpha+1}$ , for some uniform (independent of  $\alpha$ ) constant  $C > 0$ . Hence choosing  $\alpha > 0$  small enough, we get that the above quantity does not vanish, hence  $h$  is a diffeomorphism onto its image.

After this step is done, recall that we have taken coordinates given by an orthonormal basis  $\{e_i\}$  on the origin  $\mathbb{R}^{2n}$ , and by the orthonormal basis  $\{L(e_i)/\lambda_i\}$  on the target  $\mathbb{R}^{2n}$ . Written with respect to the same coordinates, we have an orthogonal transformation  $M: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  so that  $h(x) = M$  on  $B_{0.4\epsilon}(0)$ . The final step is to change the isometry  $M \in \text{SO}(2n)$  by the identity. Take a smooth path  $M_t$  of matrices joining  $M_0 = \text{Id}$  with  $M_1 = M$ . Take a smooth non-decreasing  $\rho_3: [0, \epsilon] \rightarrow [0, 1]$  with  $\rho_3(t) = 0$  for  $t \in [0, 0.2\epsilon]$ , and  $\rho_3(t) = 1$  for  $t \in [0.3\epsilon, \epsilon]$ . The map  $F(x) = M_{\rho_3(|x|)}(x)$ ,  $|x| \leq 0.4\epsilon$ , and  $F(x) = h(x)$  for  $|x| \geq 0.4\epsilon$ , is the required map.  $\square$

**Remark 3.2.** Let  $F: (\tilde{X}_c, \tilde{J}) \rightarrow (\tilde{X}_s, \tilde{\omega})$  be the diffeomorphism provided by Theorem 3.1. Then if we denote  $\tilde{\omega}' = F^*\tilde{\omega}$ , we have that  $\tilde{X}_c$  admits a symplectic structure  $\tilde{\omega}'$  and a complex

structure  $\tilde{J}$ . These are not compatible in general, but they are compatible on a neighbourhood of the exceptional locus, and give a Kähler structure there.

#### 4. A COMPLEX AND SYMPLECTIC 6-ORBIFOLD

Consider the complex Heisenberg group  $G$ , that is, the complex nilpotent Lie group of (complex) dimension 3 consisting of matrices of the form

$$\begin{pmatrix} 1 & u_2 & u_3 \\ 0 & 1 & u_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

In terms of the natural (complex) coordinate functions  $(u_1, u_2, u_3)$  on  $G$ , we have that the complex 1-forms  $\mu = du_1$ ,  $\nu = du_2$  and  $\theta = du_3 - u_2 du_1$  are left invariant, and

$$d\mu = d\nu = 0, \quad d\theta = \mu \wedge \nu.$$

Let  $\Lambda \subset \mathbb{C}$  be the lattice generated by 1 and  $\zeta = e^{2\pi i/6}$ , and consider the discrete subgroup  $\Gamma \subset G$  formed by the matrices in which  $u_1, u_2, u_3 \in \Lambda$ . We define the compact (parallelizable) nilmanifold

$$M = \Gamma \backslash G.$$

We can describe  $M$  as a principal torus bundle

$$T^2 = \mathbb{C}/\Lambda \hookrightarrow M \rightarrow T^4 = (\mathbb{C}/\Lambda)^2$$

by the projection  $(u_1, u_2, u_3) \mapsto (u_1, u_2)$ .

Consider the action of the finite group  $\mathbb{Z}_6$  on  $G$  given by the generator

$$\begin{aligned} \rho: G &\rightarrow G \\ (u_1, u_2, u_3) &\mapsto (\zeta^4 u_1, \zeta u_2, \zeta^5 u_3). \end{aligned}$$

This action satisfies that  $\rho(p \cdot q) = \rho(p) \cdot \rho(q)$ , for  $p, q \in G$ , where  $\cdot$  denotes the natural group structure of  $G$ . Moreover,  $\rho(\Gamma) = \Gamma$ . Thus,  $\rho$  induces an action on the quotient  $M = \Gamma \backslash G$ . Denote by  $\rho: M \rightarrow M$  the  $\mathbb{Z}_6$ -action. The action on 1-forms is given by

$$\rho^* \mu = \zeta^4 \mu, \quad \rho^* \nu = \zeta \nu, \quad \rho^* \theta = \zeta^5 \theta.$$

**Proposition 4.1.**  $\widehat{M} = M/\mathbb{Z}_6$  is a 6-orbifold admitting complex and symplectic structures.

*Proof.* The nilmanifold  $M$  is a complex manifold whose complex structure  $J$  is the multiplication by  $i$  at each tangent space  $T_p M$ ,  $p \in M$ . Then one can check that  $J$  commutes with the  $\mathbb{Z}_6$ -action  $\rho$  on  $M$ , that is,  $(\rho_*)_p \circ J_p = J_{\rho(p)} \circ (\rho_*)_p$ , for any point  $p \in M$ . Hence,  $J$  induces a complex structure on the quotient  $\widehat{M} = M/\mathbb{Z}_6$ .

Now we define the complex 2-form  $\omega$  on  $M$  given by

$$(3) \quad \omega = -i \mu \wedge \bar{\mu} + \nu \wedge \theta + \bar{\nu} \wedge \bar{\theta}.$$

Clearly,  $\omega$  is a real closed 2-form on  $M$  which satisfies  $\omega^3 > 0$ , that is,  $\omega$  is a symplectic form on  $M$ . Moreover,  $\omega$  is  $\mathbb{Z}_6$ -invariant. Indeed,  $\rho^* \omega = -i \mu \wedge \bar{\mu} + \zeta^6 \nu \wedge \theta + \zeta^{-6} \bar{\nu} \wedge \bar{\theta} = \omega$ . Therefore  $\widehat{M}$  is a symplectic 6-orbifold, with the symplectic form  $\widehat{\omega}$  induced by  $\omega$ .  $\square$

We denote by

$$\widehat{\pi}: M \rightarrow \widehat{M}$$

the natural projection. The orbifold points of  $\widehat{M}$  are the following:

- (1) The points  $(\frac{1}{3}a(1+\zeta), \frac{1}{3}b(1+\zeta), \frac{1}{3}c(1+\zeta) + \frac{2}{9}ab(1+\zeta)^2) \in M$ ,  $a, b, c \in \{0, 1, 2\}$  and  $(b, c) \neq (0, 0)$ , are points of order 3, with isotropy group  $K = \{\text{Id}, \rho^2, \rho^4\}$ . These points are mapped in pairs by  $\mathbb{Z}_6$ , so they define 12 orbifold points in  $\widehat{M} = M/\mathbb{Z}_6$ , with models  $\mathbb{C}^3/K$ .
- (2) The surfaces  $S_{(p,q)} = \{(u_1, p, p u_1 + q) \mid u_1 \in \mathbb{C}/\Lambda\} \subset M$ , where  $p, q \in \{0, \frac{1}{2}, \frac{\zeta}{2}, \frac{1+\zeta}{2}\}$ ,  $(p, q) \neq (0, 0)$ . These are 15 tori, which consist of points of order 2, with isotropy  $H = \{\text{Id}, \rho^3\}$ . These surfaces are permuted by the group  $\mathbb{Z}_6$ , so they come in 5 groups of three tori each. Thus they define 5 tori in the orbifold  $\widehat{M}$ , formed by orbifold points of order 2.
- (3) The surface  $S_0 = \{(u_1, 0, 0) \mid u_1 \in \mathbb{C}/\Lambda\} \subset M$  is a torus consisting generically of points of order 2, with isotropy  $H$ . Here  $\rho: S_0 \rightarrow S_0$  and it is a map of order 3, with three fixed points  $(\frac{1}{3}a(1+\zeta), 0, 0)$ ,  $a = 0, 1, 2$ . These points have isotropy  $\mathbb{Z}_6$ . The quotient  $S_0/\langle\rho\rangle \subset \widehat{M}$  is homeomorphic to a sphere (with three orbifold points of order 3).

## 5. RESOLUTION OF THE 6-ORBIFOLD

Now we want to desingularize the orbifold  $\widehat{M}$ . We shall treat each of the connected components of the singular locus determined before independently. Recall that  $K = \{\text{Id}, \rho^2, \rho^4\} \cong \mathbb{Z}_3$  and  $H = \{\text{Id}, \rho^3\} \cong \mathbb{Z}_2$ . There is a natural isomorphism  $\langle\rho\rangle = \mathbb{Z}_6 \cong K \times H$ .

**5.1. Resolution of the isolated orbifold points.** We know that there are 12 isolated orbifold points in  $\widehat{M}$ . Let  $\widehat{p} \in \widehat{M}$  be one of them. The preimage of  $\widehat{p}$  under  $\widehat{\pi}$  consists of two points,  $\widehat{\pi}^{-1}(\widehat{p}) = \{p_1, p_2\}$ . The isotropy group of  $p_1$  is  $K$ . Consider a  $K$ -invariant neighbourhood  $U$  of  $p_1$  in  $M$ . Then,

$$\widehat{U} = \widehat{\pi}(U) \cong U/K$$

is an orbifold neighbourhood of  $\widehat{p}$  in  $\widehat{M}$ . This has complex and symplectic resolutions as in Section 3. In order to apply Theorem 3.1 we check that  $\iota = \iota': K \rightarrow \text{U}(3)$ . For the complex resolution, we have  $\iota(\zeta^2) = \text{diag}(\zeta^2, \zeta^2, \zeta^4)$ . For the symplectic resolution, the symplectic form (3) is, in our coordinates  $(u_1, u_2, u_3)$ ,

$$(4) \quad \omega = -i du_1 \wedge d\bar{u}_1 + du_2 \wedge du_3 + d\bar{u}_2 \wedge d\bar{u}_3.$$

We have to do a change of variables to transform  $K \subset \text{Sp}(6, \mathbb{R})$  into a subgroup of  $\text{U}(3)$ . This is obtained with

$$\begin{aligned} v_1 &= u_1 \\ v_2 &= \frac{1}{\sqrt{2}}(u_2 - i\bar{u}_3) \\ v_3 &= \frac{1}{\sqrt{2}}(\bar{u}_2 - iu_3). \end{aligned}$$

This transforms (4) into

$$\omega = -i dv_1 \wedge d\bar{v}_1 - i dv_2 \wedge d\bar{v}_2 - i dv_3 \wedge d\bar{v}_3,$$

the standard Kähler form. The  $K$ -action is given by  $(v_1, v_2, v_3) \mapsto (\zeta^2 v_1, \zeta^2 v_2, \zeta^4 v_3)$ , so  $\iota'(\zeta^2) = \text{diag}(\zeta^2, \zeta^2, \zeta^4)$ , and  $\iota = \iota'$ .



**5.2. Resolution of the singular sets  $\widehat{\pi}(S_{(p,q)})$ .** Now we consider a connected component of the singular set which is homeomorphic to a 2-torus. There are 5 such components in  $\widehat{M}$ , all of them are images by  $\widehat{\pi}$  of the sets  $S_{(p,q)} = \{(u_1, p, p u_1 + q) \mid u_1 \in \mathbb{C}/\Lambda\}$ , where  $(p, q) \in I = \left(\{0, \frac{1}{2}, \frac{\zeta}{2}, \frac{1+\zeta}{s}\}\right)^2 - \{(0, 0)\}$ .

Let us focus on one such component  $\widehat{T} = \widehat{\pi}(T)$ ,  $T \cong \mathbb{C}/\Lambda$ . Then  $H$  fixes  $S_{(p,q)}$ , and its orbit under  $K$  is given by  $S_{(p_i, q_i)}$ , for three elements  $(p_1, q_1) = (p, q), (p_2, q_2), (p_3, q_3) \in I$ . Consider a neighbourhood  $U$  of  $T \subset M$  via

$$\begin{aligned} T \times B_\epsilon(0) &\rightarrow U \\ (u_1, u_2, u_3) &\mapsto (u_1, u_2 + p, u_3 + p u_1 + q), \end{aligned}$$

where  $B_\epsilon(0) \subset \mathbb{C}^2$ . The image is

$$(5) \quad \widehat{U} = \widehat{\pi}(U) \cong U/H \cong T \times (B_\epsilon(0)/H),$$

where  $H \cong \mathbb{Z}_2$  acts as  $(u_2, u_3) \mapsto (-u_2, -u_3)$ .

We see that the complex structure on (5) is the product complex structure. Also, the symplectic structure  $\omega = i du_1 \wedge d\bar{u}_1 + du_2 \wedge du_3 + d\bar{u}_2 \wedge d\bar{u}_3$  is the product of the natural symplectic structure of  $\mathbb{C}/\Lambda$  with an orbifold symplectic structure on  $B_\epsilon(0)/H$ . Using the construction of Section 3, we have a desingularization

$$\widetilde{Y} \rightarrow B_\epsilon(0)/H$$

which is a smooth manifold endowed with both a complex structure and a symplectic structure coinciding with the given ones outside a small neighbourhood of the exceptional locus  $E$ . The condition  $\iota = \iota'$  of Theorem 3.1 is trivially satisfied, since  $\iota(\rho^3) = \iota'(\rho^3) = -\text{Id}$ . Multiplying by  $T = \mathbb{C}/\Lambda$ , we have that

$$\widetilde{U} = T \times \widetilde{Y}$$

is a smooth manifold endowed with a complex structure  $\widetilde{J}$ , and a symplectic structure  $\widetilde{\omega}$ , which coincide with those of  $\widehat{U}$  outside a small neighbourhood of the exceptional locus  $T \times E \subset \widetilde{U}$ .

The complex and the symplectic resolutions of  $\widehat{M}$  in a neighbourhood of  $\widehat{T}$  are obtained by replacing  $\widehat{U} \subset \widehat{M}$  with  $\widetilde{U}$ . The two resolutions are diffeomorphic by the considerations above.

**5.3. Resolution of the singular set  $\widehat{\pi}(S_0)$ .** Finally we consider the connected component of the singular set which is homeomorphic to a 2-sphere. This is  $\widehat{S}_0 = \widehat{\pi}(S_0)$ , where  $S_0 = \{(u_1, 0, 0) \mid u_1 \in \mathbb{C}/\Lambda\}$ . As before, a neighbourhood of  $S_0$  in  $M$  is of the form

$$U_0 = (\mathbb{C}/\Lambda) \times B_\epsilon(0),$$

where  $B_\epsilon(0) \subset \mathbb{C}^2$ . The action of  $H = \mathbb{Z}_2$  is trivial on  $\mathbb{C}/\Lambda$  and as  $\pm 1$  on  $\mathbb{C}^2$ . The action of  $K = \mathbb{Z}_3$  is of the form  $\rho^2(u_1, u_2, u_3) = (\zeta^2 u_1, \zeta^2 u_2, \zeta^4 u_3)$ .

Let us focus on  $B_\epsilon(0)/H$ . By the construction of Section 3, we have a complex desingularization  $(\widetilde{Y}_c, \widetilde{J}) \rightarrow B_\epsilon(0)/H$ . The holomorphic action of  $K$  on  $B_\epsilon(0)$  induces an action on  $(\widetilde{Y}_c, \widetilde{J})$ . Also, there is a symplectic desingularization  $(\widetilde{Y}_s, \widetilde{\omega}) \rightarrow B_\epsilon(0)/H$ . The action of  $K$  on  $B_\epsilon(0)$  induces an action on  $(\widetilde{Y}_s, \widetilde{\omega})$ . This follows by taking an orbifold chart of the singular point that is  $(H \times K)$ -equivariant, using the equivariant Darboux theorem.

By Theorem 3.1, there is a diffeomorphism  $F: (\tilde{Y}_c, \tilde{J}) \rightarrow (\tilde{Y}_s, \tilde{\omega})$ . Let us see that  $F$  can be taken to be  $K$ -equivariant. This follows by the arguments in the proof of Theorem 3.1 by using that  $\iota: H \times K \rightarrow \mathrm{U}(2)$  and  $\iota': H \times K \rightarrow \mathrm{U}(2)$  are equal. For the complex case,  $\iota$  is given by the representation  $(u_2, u_3) \mapsto (\zeta u_2, \zeta^5 u_3)$ , so  $\iota(\zeta) = \mathrm{diag}(\zeta, \zeta^5)$ . For the symplectic case, we have to do a change of variables to transform  $H \times K \subset \mathrm{Sp}(4, \mathbb{R})$  into a subgroup of  $\mathrm{U}(2)$ . This is given by

$$v_2 = \frac{1}{\sqrt{2}}(u_2 - i\bar{u}_3), \quad v_3 = \frac{1}{\sqrt{2}}(\bar{u}_2 - iu_3),$$

which transforms  $\omega = du_2 \wedge du_3 + d\bar{u}_2 \wedge d\bar{u}_3$  into the standard Kähler form  $-i dv_2 \wedge d\bar{v}_2 - i dv_3 \wedge d\bar{v}_3$ . As  $(v_2, v_3) \mapsto (\zeta v_2, \zeta^5 v_3)$ , we have that  $\iota'(\zeta) = \mathrm{diag}(\zeta, \zeta^5)$ . Hence  $\iota = \iota'$ .

This produces a desingularization  $\tilde{Y} \rightarrow B_\epsilon(0)/H$  with a symplectic and a complex structure, which match the given ones outside a small neighbourhood of the exceptional set  $E \subset \tilde{Y}$ , which are compatible (they give a Kähler structure) in a smaller neighbourhood of  $E$ , by Remark 3.2, and which have an action of  $K$  preserving both the complex and symplectic structures. A desingularization of

$$U_0/H = (\mathbb{C}/\Lambda) \times (B_\epsilon(0)/H)$$

is given by substituting a neighbourhood of  $\hat{S}_0 = (\mathbb{C}/\Lambda) \times \{0\}$  by  $(\mathbb{C}/\Lambda) \times \tilde{Y}$ . The fixed points of action of  $K$  in  $U_0/H$  lie on  $\hat{S}_0$ , hence the fixed points of the action of  $K$  on the desingularization of  $U_0/H$  lie in the exceptional divisor. In this part of the manifold, we have a Kähler structure, so the symplectic and complex desingularization are the same.

This means that  $(U_0/H)/K \cong U_0/(H \times K)$  admits a desingularization  $\tilde{V}$  with a complex and a symplectic structure. The resolution of  $\widehat{M}$  in a neighbourhood of  $\hat{S}_0$  is obtained by substituting  $\widehat{\pi}(U_0) = U_0/(H \times K) \subset \widehat{M}$  with  $\tilde{V}$ .

All together, we get a smooth 6-manifold  $\widetilde{M}$  with a complex structure and a symplectic structure, and with a map

$$\pi: \widetilde{M} \longrightarrow \widehat{M},$$

which is simultaneously a complex and a symplectic resolution.

## 6. TOPOLOGICAL PROPERTIES OF $\widetilde{M}$

In this section, we are going to complete the proof of Theorem 1.1 by proving that  $\widetilde{M}$  is simply-connected and that it does not admit a Kähler structure.

**Proposition 6.1.**  *$\widetilde{M}$  is simply connected.*

*Proof.* We fix base points  $p_0 = (0, 0, 0) \in M$  and  $\widehat{p}_0 = \widehat{\pi}(p_0) \in \widehat{M}$ . There is an epimorphism of fundamental groups

$$\Gamma = \pi_1(M, p_0) \twoheadrightarrow \pi_1(\widehat{M}, \widehat{p}_0),$$

since the  $\mathbb{Z}_6$ -action has a fixed point [4, Chapter II, Corollary 6.3]. Now the nilmanifold  $M$  is a principal 2-torus bundle over the 4-torus  $T^4$ , so we have an exact sequence

$$\mathbb{Z}^2 \hookrightarrow \Gamma \rightarrow \mathbb{Z}^4.$$

The group  $\Gamma = \pi_1(M, p_0)$  is thus generated by the images of the fundamental groups of the surfaces  $\Sigma_1 = \{(u_1, 0, 0)\}$ ,  $\Sigma_2 = \{(0, u_2, 0)\}$  and  $\Sigma_3 = \{(0, 0, u_3)\}$  in  $M$ . The image  $\widehat{\pi}(\Sigma_1)$  is a 2-sphere, since  $\widehat{\pi}: \Sigma_1 \rightarrow \widehat{\pi}(\Sigma_1)$  is a degree 3 map with three ramification points

of order 3 (namely  $(\frac{1}{2}a(1+\zeta), 0, 0)$ , with  $a = 0, 1, 2$ ). The image of  $\Sigma_2$  is also a 2-sphere, since  $\hat{\pi}: \Sigma_2 \rightarrow \hat{\pi}(\Sigma_2)$  is a degree 6 map with one point of order 6,  $(0, 0, 0)$ , two of order 3,  $(0, \frac{1}{2}b(1+\zeta), 0)$ ,  $b = 1, 2$ , and three of order 2 (namely  $(0, p, 0)$ ,  $p = \frac{1}{2}, \frac{\zeta}{2}, \frac{1+\zeta}{2}$ ). Analogously,  $\hat{\pi}(\Sigma_3)$  is a 2-sphere. This proves that  $\pi_1(\widehat{M}, \widehat{p}_0) = \{1\}$ .

Now we look at the resolution process. Let  $S \subset \widehat{M}$  be the singular locus and suppose  $p \in S$  is an isolated orbifold point. The resolution replaces a neighbourhood  $B = B_\epsilon(0)/\Gamma_p$  of  $p$  with a smooth manifold  $\widetilde{B}$ , such that  $\pi: \widetilde{B} \rightarrow B$  is a complex resolution of singularities. The manifold  $\widetilde{B}$  is simply connected by [22, Theorem 4.1]. A Seifert-Van Kampen argument gives that  $\pi_1(\widehat{M})$  is the amalgamated sum of  $\pi_1(\widehat{M} - \{p\})$  and  $\pi_1(B)$  along  $\pi_1(\partial B)$ . Also  $\pi_1(\widehat{M})$  is the amalgamated sum of  $\pi_1(\widehat{M} - E)$  and  $\pi_1(\widetilde{B})$  along  $\pi_1(\partial B)$ . As  $\pi_1(B) = \pi_1(\widetilde{B}) = \{1\}$ , we have that  $\pi_1(\widehat{M}) = \pi_1(\widetilde{M})$ .

Suppose now that we have a connected component  $S'$  of the singular locus  $S$  of positive dimension. Let  $E' = \pi^{-1}(S')$  be the corresponding exceptional locus. The invariance of the fundamental group under resolution is proved along the same lines as before if we know that the map  $\pi: E' \rightarrow S'$  induces an isomorphism  $\pi_1(E') \rightarrow \pi_1(S')$ . In our case, we have two possibilities: if  $S' = \hat{\pi}(S_{(p,q)}) \cong T^2$ , then  $E' = T^2 \times E$ , where  $E$  is the exceptional divisor of the resolution  $\widetilde{Y} \rightarrow B_\epsilon(0)/H$ , which is clearly simply connected, and the result follows.

The second possibility is  $S' = \hat{\pi}(S_0)$ . In this case, the exceptional divisor over  $S'$  is the exceptional divisor of the resolution of

$$((\mathbb{C}/\Lambda) \times (\mathbb{C}^2/H))/K.$$

The resolution of  $\mathbb{C}^2/H$  is done by blowing-up  $\mathbb{C}^2$  at the origin,

$$\widetilde{\mathbb{C}}^2 = \{(a, b, [u : v]) \in \mathbb{C}^2 \times \mathbb{CP}^1 \mid av = bu\},$$

and then quotienting by  $H = \{\pm \text{Id}\}$ . Clearly, the fundamental groups of  $(\mathbb{C}/\Lambda) \times (\mathbb{C}^2/H)$  and  $(\mathbb{C}/\Lambda) \times (\widetilde{\mathbb{C}}^2/H)$  coincide. The action of  $K$  is given by  $(a, b, [u : v]) \mapsto ((\zeta^2 a, \zeta^4 b), [u : \zeta^2 v])$ , with fixed points  $(0, 0, [1 : 0])$  and  $(0, 0, [0 : 1])$ . The fixed points of  $K$  on  $((\mathbb{C}/\Lambda) \times (\widetilde{\mathbb{C}}^2/H))$  occur when  $K$  fixes both factors. Therefore, all fixed points are isolated, and the second resolution does not alter the fundamental group.  $\square$

In order to prove that  $\widetilde{M}$  does not admit a Kähler structure, we are going to check that it does not satisfy the Lefschetz condition *for any symplectic form*. For this, it is necessary to understand the cohomology  $H^*(\widetilde{M})$ .

We start by computing the cohomology of  $\widehat{M}$ . By Nomizu theorem [18], the cohomology of the nilmanifold  $M$  is:

$$\begin{aligned} H^0(M, \mathbb{C}) &= \langle 1 \rangle, \\ H^1(M, \mathbb{C}) &= \langle [\mu], [\bar{\mu}], [\nu], [\bar{\nu}] \rangle, \\ H^2(M, \mathbb{C}) &= \langle [\mu \wedge \bar{\mu}], [\mu \wedge \bar{\nu}], [\bar{\mu} \wedge \nu], [\nu \wedge \bar{\nu}], [\mu \wedge \theta], [\bar{\mu} \wedge \bar{\theta}], [\nu \wedge \theta], [\bar{\nu} \wedge \bar{\theta}] \rangle, \\ H^3(M, \mathbb{C}) &= \langle [\mu \wedge \bar{\mu} \wedge \theta], [\mu \wedge \bar{\mu} \wedge \bar{\theta}], [\nu \wedge \bar{\nu} \wedge \theta], [\nu \wedge \bar{\nu} \wedge \bar{\theta}], [\mu \wedge \nu \wedge \theta], [\bar{\mu} \wedge \bar{\nu} \wedge \bar{\theta}] \\ &\quad [\mu \wedge \bar{\nu} \wedge \theta], [\mu \wedge \bar{\nu} \wedge \bar{\theta}], [\bar{\mu} \wedge \nu \wedge \theta], [\bar{\mu} \wedge \nu \wedge \bar{\theta}] \rangle, \\ H^4(M, \mathbb{C}) &= \langle [\mu \wedge \bar{\mu} \wedge \nu \wedge \theta], [\mu \wedge \bar{\mu} \wedge \bar{\nu} \wedge \bar{\theta}], [\bar{\mu} \wedge \nu \wedge \bar{\nu} \wedge \bar{\theta}], [\mu \wedge \nu \wedge \bar{\nu} \wedge \theta], \\ &\quad [\mu \wedge \bar{\mu} \wedge \theta \wedge \bar{\theta}], [\nu \wedge \bar{\nu} \wedge \theta \wedge \bar{\theta}], [\mu \wedge \bar{\nu} \wedge \theta \wedge \bar{\theta}], [\bar{\mu} \wedge \nu \wedge \theta \wedge \bar{\theta}] \rangle, \end{aligned}$$

$$H^5(M, \mathbb{C}) = \langle [\mu \wedge \bar{\mu} \wedge \nu \wedge \theta \wedge \bar{\theta}], [\mu \wedge \bar{\mu} \wedge \bar{\nu} \wedge \theta \wedge \bar{\theta}], [\mu \wedge \nu \wedge \bar{\nu} \wedge \theta \wedge \bar{\theta}], [\bar{\mu} \wedge \nu \wedge \bar{\nu} \wedge \theta \wedge \bar{\theta}] \rangle,$$

$$H^6(M, \mathbb{C}) = \langle [\mu \wedge \bar{\mu} \wedge \nu \wedge \bar{\nu} \wedge \theta \wedge \bar{\theta}] \rangle.$$

So the cohomology  $H^*(\widehat{M}) = H^*(M)^{\mathbb{Z}_6}$  of  $\widehat{M}$  is:

$$\begin{aligned} H^0(\widehat{M}, \mathbb{C}) &= \langle 1 \rangle, \\ H^1(\widehat{M}, \mathbb{C}) &= 0, \\ H^2(\widehat{M}, \mathbb{C}) &= \langle [\mu \wedge \bar{\mu}], [\nu \wedge \bar{\nu}], [\nu \wedge \theta], [\bar{\nu} \wedge \bar{\theta}] \rangle, \\ H^3(\widehat{M}, \mathbb{C}) &= 0, \\ H^4(\widehat{M}, \mathbb{C}) &= \langle [\mu \wedge \bar{\mu} \wedge \nu \wedge \theta], [\mu \wedge \bar{\mu} \wedge \bar{\nu} \wedge \bar{\theta}], [\mu \wedge \bar{\mu} \wedge \theta \wedge \bar{\theta}], [\nu \wedge \bar{\nu} \wedge \theta \wedge \bar{\theta}] \rangle, \\ H^5(\widehat{M}, \mathbb{C}) &= 0, \\ H^6(\widehat{M}, \mathbb{C}) &= \langle [\mu \wedge \bar{\mu} \wedge \nu \wedge \bar{\nu} \wedge \theta \wedge \bar{\theta}] \rangle. \end{aligned}$$

**Proposition 6.2.**  *$\widetilde{M}$  does not admit a Kähler structure since it does not satisfy the Lefschetz property for any symplectic form on  $\widetilde{M}$ .*

*Proof.* Let  $\Omega$  be a symplectic form on  $\widetilde{M}$ . The Lefschetz map  $L_{[\Omega]}: H^2(\widetilde{M}) \rightarrow H^4(\widetilde{M})$  is given by the cup product with  $[\Omega]$ . We show that there is a class  $[\beta] \in H^2(\widetilde{M})$  which is in the kernel of  $L_{[\Omega]}$ . We prove this by checking that  $[\Omega] \wedge [\beta] \wedge [\alpha] = 0$ , for any 2-form  $[\alpha] \in H^2(\widetilde{M})$ .

We need to determine the cohomology  $H^2(\widetilde{M})$ . For this, the first step is to construct a map  $H^2(\widetilde{M}) \rightarrow H^2(\widehat{M})$ . Let  $h: M \rightarrow M$  be a map which:

- is the identity outside small neighbourhoods of each point with non-trivial isotropy,
- contracts a neighbourhood of each of the isolated 24 points with isotropy  $K$  onto the corresponding point,
- contracts a neighbourhood of each  $S_{(p,q)}$  onto  $S_{(p,q)}$  (fixing  $S_{(p,q)}$  pointwise),
- in a neighbourhood of  $S_0$ , is the composition of a contraction onto  $S_0$  with a map that contracts neighbourhoods (in  $S_0$ ) of the 3 fixed points to the points, and
- is  $\mathbb{Z}_6$ -equivariant.

$h$  induces a map  $\widehat{h}: \widehat{M} \rightarrow \widehat{M}$ . Note that for any closed form  $\alpha \in \Omega^*(\widehat{M})$ ,  $\widehat{h}^*(\alpha) \in \Omega^*(\widehat{M})$  is cohomologous to  $\alpha$  and can be lifted to a form  $\pi^* \widehat{h}^*(\alpha) \in \Omega^*(\widetilde{M})$ , where  $\pi: \widetilde{M} \rightarrow \widehat{M}$  is the resolution map. This induces a well-defined map

$$\Psi = \pi^* \circ \widehat{h}^*: H^*(\widehat{M}) \rightarrow H^*(\widetilde{M}).$$

Now consider  $U = \widehat{M} - S$ , where  $S \subset \widehat{M}$  is the singular locus and  $V \subset \widehat{M}$  is a small neighbourhood of  $S$ . Let also  $\widetilde{U} = \pi^{-1}(U)$  and  $\widetilde{V} = \pi^{-1}(V) \subset \widetilde{M}$ . Using compactly supported de Rham cohomology, we have a diagram

$$\begin{array}{ccccccc} H_c^2(U) \oplus H_c^2(V) & \rightarrow & H_c^2(\widehat{M}) & \rightarrow & H_c^3(U \cap V) & \rightarrow & H_c^3(U) \oplus H_c^3(V) \\ \downarrow = & \Psi \downarrow & \downarrow \Psi & & \downarrow \cong & & \downarrow = \Psi \downarrow \\ H_c^2(\widetilde{U}) \oplus H_c^2(\widetilde{V}) & \rightarrow & H_c^2(\widetilde{M}) & \rightarrow & H_c^3(\widetilde{U} \cap \widetilde{V}) & \rightarrow & H_c^3(\widetilde{U}) \oplus H_c^3(\widetilde{V}) \end{array}$$

Since  $V$  retracts onto a set of dimension 2,  $H^3(V) = 0$ . By Poincaré duality,  $H_c^3(V) = 0$  as well. Now a simple diagram chasing proves that  $H^2(\widetilde{M}) = H_c^2(\widetilde{M})$  is generated by  $H^2(\widehat{M}) = H_c^2(\widehat{M})$  and  $H_c^2(\widetilde{V})$ .

Consider the closed form  $\nu \wedge \bar{\nu} \in \Omega^2(\widehat{M})$ . Since  $\nu \wedge \bar{\nu}|_{S_{(p,q)}} = 0$  for any surface  $S_{(p,q)}$  and  $\nu \wedge \bar{\nu}|_{S_0} = 0$  as well, the 2-cohomology class

$$[\beta] = \Psi([\nu \wedge \bar{\nu}])$$

vanishes on  $\widetilde{V}$ . Clearly  $[\beta] \wedge [\alpha_1] \wedge [\alpha_2] = 0$  if either  $[\alpha_1], [\alpha_2] \in H_c^2(\widetilde{V})$ . Moreover, one can check that  $[\beta] \wedge [\alpha_1] \wedge [\alpha_2] = 0$ , for  $[\alpha_1], [\alpha_2] \in H^2(\widehat{M})$ , which completes the proof.  $\square$

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